Some recent results on edge-colored graphs

Shinya Fujita (Yokohama City University)

☆ The topic is based on the following joint papers with my Chinese colleagues.

- "Color degree and monochromatic degree conditions for short properly colored cycles in edge-colored graphs " JGT 2018 (with Ruonan Li and Shinggui Zhang)
- "On sufficient conditions for rainbow cycles in edge-colored graphs" DM, accepted (with Bo Ning, Chuandong Xu and Shenggui Zhang)
- "Decomposing edge-colored graphs under color degree constraints" CPC, accepted (with Ruonan Li and Guanghui Wang)

Part I: Degree results

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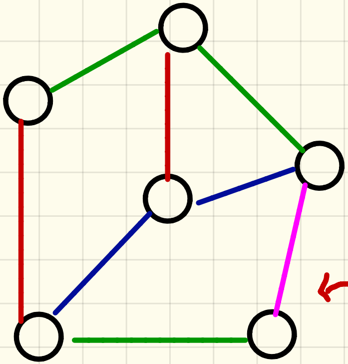
In this talk, we consider degree condition for cycles in edge-colored graphs. Let

$$\delta^c(G) := \min\{d^c(v) \mid v \in V(G)\}$$

$d^c(v)$: color degree of v; i.e., the number of colors adjacent to v in G.

$$\boxed{\text{Note: } \delta(G) \geq \delta^c(G)}$$

Ex. G



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properly colored C_4!

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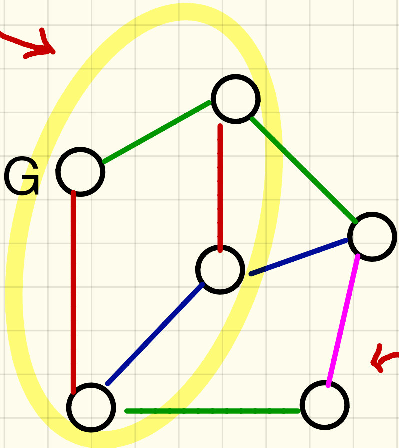

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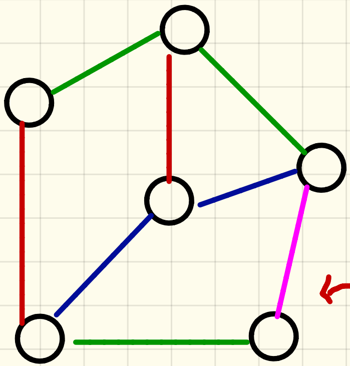

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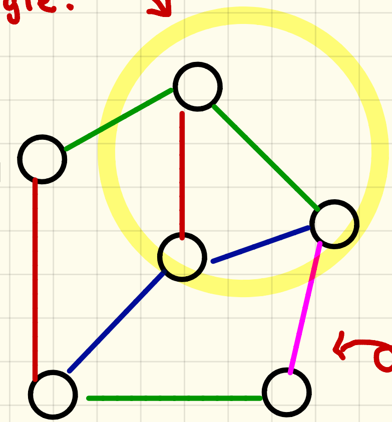
color degree of v; i.e., the number of colors adjacent to v in G.

rainbow triangle!

Ex. G

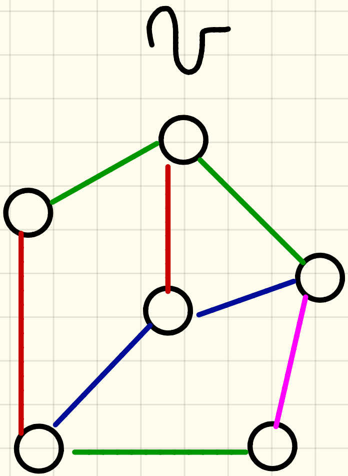


$\delta^c(G) = 2$

$\boxed{\text{Note: } \delta(G) \geq \delta^c(G)}$

For a vertex v in an edge-colored graph G, let $CN(v)$ be the set of colors assigned to edges incident to v.

Ex. G



$CN(v) = \{$ green, red $\}$

☆ Some natural questions:

What is the sharp degree conditions for the followings?

Prop. 1: If G is an edge-colored graph of order n with $\delta^c(G) \geq f(n)$, then G contains a properly colored cycle.

Prop. 2: If G is an edge-colored graph of order n with $\delta^c(G) \geq g(n)$, then G contains a rainbow cycle.

☆ Answer for Prop.1

Prop. 1: If G is an edge-colored graph of order n with $\delta^c(G) \geq f(n)$, then G contains a properly colored cycle.

Th 1 (Li, Zhang and F, JGT 2018)

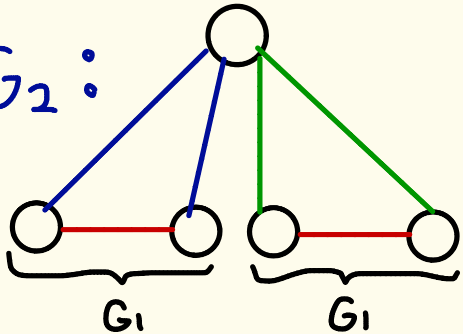
Let D be the least value of $f(n)$ s.t. Prop.1 is true.

Then $n+1 = D! \sum_{i=0}^{D} \frac{1}{i!}$ holds.

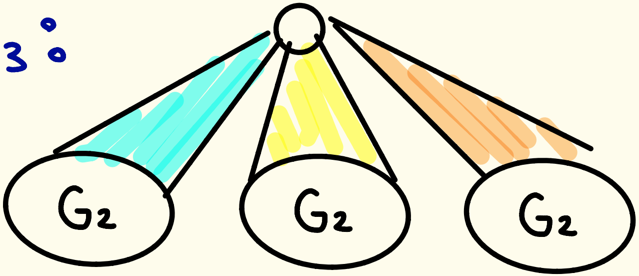
Construction of sharpness example:

G_1:

G_2:



G_3:



Doing this way, we can construct G_{i+1} from G_i so that $\delta^c(G_{i+1}) = i+1$ and G_{i+1} has no PC cycle.

Note: $\delta^c(G_D) = D$,

$$|V(G_D)| = D! \sum_{i=0}^{D} \frac{1}{i!}$$

☆ Partial answer for Prop. 2

Prop. 2: If G is an edge-colored graph of order n with $\delta^c(G) \geq g(n)$, then G contains a rainbow cycle.

Th 2 (Li et al. EuJC 2014)

Let D be the least value of $g(n)$ s.t. Prop. 2 is true. Then $D < \frac{n}{2} + 1$ holds.

Th 2 (Li et al. EuJC 2014)

Let G be an edge-colored graph of order $n \geq 5$ with $\delta^c(G) \geq \frac{n}{2}$. Then $G \supset \exists$ rainbow triangle or $G \simeq K_{\frac{n}{2},\frac{n}{2}}$.

Th 3 (Broersma et al. AuJC 2005)

Let G be an edge-colored graph of order $n \geq 4$ s.t. $|CN(u) \cup CN(v)| \geq n-1$ for every pair $\underset{(\neq)}{u,v} \in V(G)$.

Then $G \supset \exists$ rainbow triangle or $\exists$ rainbow C_4.

Our results are following.

Th 4 (Ning, Xu, Zhang and F)

For $k \geq 1$, let G be an edge-colored graph of order $n \geq 105k - 24$ s.t. $|CN(u) \cup CN(v)| \geq n-1$ for every pair $\underset{(\neq)}{u,v} \in V(G)$.

Then $G \supset$ k rainbow C_4.

Th 5 (Ning, Xu, Zhang and F)

Let G be an edge-colored graph of order $n \geq 6$ s.t. $|CN(u) \cup CN(v)| \geq n-1$ for every pair $\underset{(\neq)}{u,v} \in V(G)$.

Then $G \supset {}^{\exists}$ rainbow triangle or $G \simeq K_{\frac{n}{2},\frac{n}{2}}$.

Our results are following.

Th 6 (Ning, Xu, Zhang and F)

For $k \geq 1$, let G be an edge-colored graph of order n s.t. $|CN(u) \cup CN(v)| \geq \frac{n}{2} + 64k + 1$ for every pair $\underset{(\neq)}{u, v} \in V(G)$.

Then $G \supset k$ vertex-disjoint rainbow cycles.

Cor. For $k \geq 1$, if G is an edge-colored graph of order n with $\delta^c(G) \geq \frac{n}{2} + 64k + 1$, then $G \supset k$ vertex-disjoint rainbow cycles.

Our results are following:

Th. 7 (Li, Zhang and F JGT 2018)

If $\delta^c(K_{m,n}) \geq 2$ then $\exists$ PC C_4 or C_6 in $K_{m,n}$.

Th. 8 (Li, Zhang and F JGT 2018)

If $\delta^c(K_{m,n}) \geq 3$ then $\exists$ PC C_4 in $K_{m,n}$.

Remark. The minimum color degree conditions are sharp.

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If $\delta^c(K_{m,n}) \geq 2$ then $^\exists$ PC C_4 or C_6 in $K_{m,n}$.

Cor.

~~Th. 8~~ (Li, Zhang and F JGT 2018)

If $\delta^c(K_{m,n}) \geq 3k$ then $^\exists$ k PC C_4 in $K_{m,n}$.

Remark. The minimum color degree conditions are sharp.

I propose the following conjecture:

Conj.

If $\delta^c(K_{m,n}) \geq \frac{m+n}{4} + 1$ then each vertex is contained in properly colored cycles of length $4, 6, \cdots, \min\{2m, 2n\}$, respectively.

We have the following partial result to this conjecture.

Th. 9 (Li, Zhang and F JGT 2018)

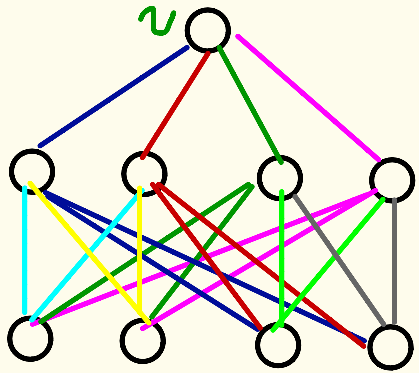
If $\delta^c(K_{m,n}) \geq \frac{m+n}{4} + 1$ then each vertex is contained in a properly colored cycle of length 4.

The bound on the color degree condition is best possible.

Prop. $\exists$ edge-coloring of $K_{m,n}$ s.t. $\delta^c(K_{m,n}) = \frac{m+n+3}{4}$ and $\exists\, v \in K_{m,n}$ s.t. any properly colored C_4 does not contain v.

The case where m=5, n=4:

$$\delta^c(K_{5,4}) = \frac{5+4+3}{4}$$



Part II: Decomposition results

- ”Decomposing edge-colored graphs under color degree constraints” CPC, accepted (with Ruonan Li and Guanghui Wang)

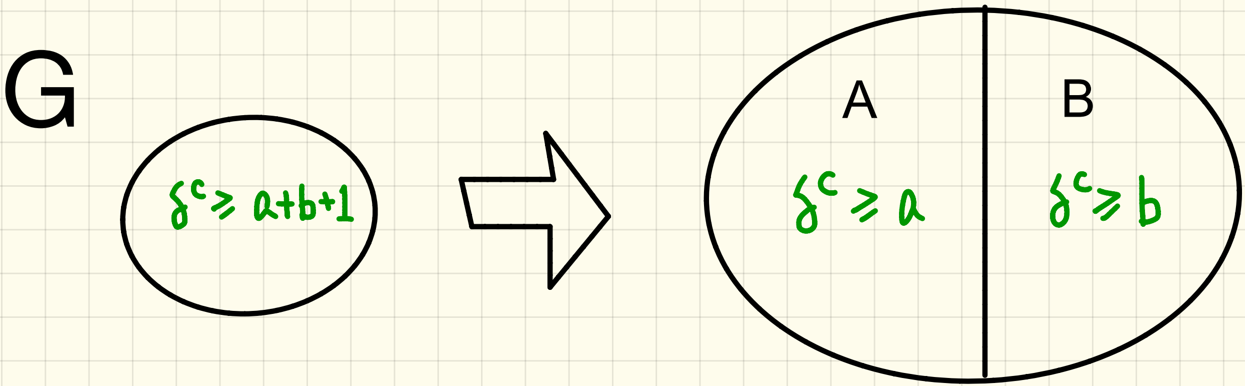
I propose the following conjecture:

Conj.

Let G be an edge-colored graph with $\delta^c(G) \geq a+b+1$.

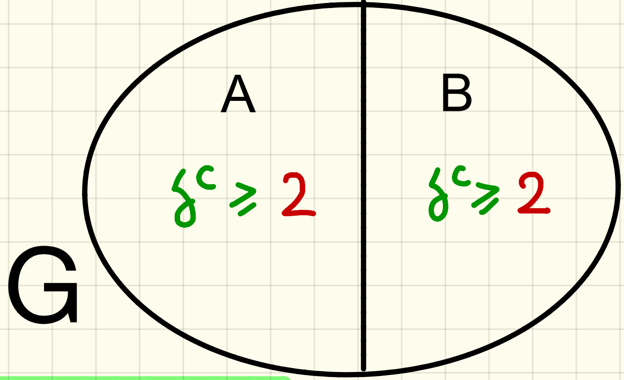
Then G can be partitioned into 2 parts A and B s.t.

$$\delta^c(G[A]) \geq a \text{ and } \delta^c(G[B]) \geq b.$$



Our main results are following.

- Conj. is true for $a=b=2$.



Thm. (Ruonan Li, Guanghui Wang, and F)

Let G be an edge-colored graph with $\delta^c(G) \geq 5$.

Then G can be partitioned into 2 parts A and B s.t.

$$\delta^c(G[A]) \geq 2 \text{ and } \delta^c(G[B]) \geq 2.$$

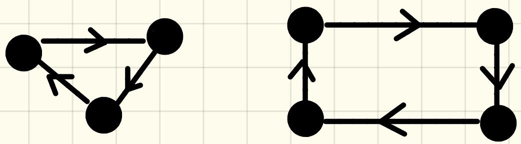
Our results are closely related to Bermond-Thomassen's conjecture in digraphs.

Pbm. Determine the least value f(k) which makes the following proposition true.

Prop. Every digraph D with $\delta^+(D) \geqslant f(k)$ contains k vertex-disjoint dicycles.

Conj. (Bermond and Thomassen, JGT'81)

$$f(k) = 2k-1.$$



$\hookrightarrow$ Known results: True for $k \leqq 3$.

In fact, we obtained a stronger statement. To state this, let g(k) be the following function.

$$g(k) = \begin{cases} 2 & (k=1) \\ \max\{ f(k)+1, g(k-1)+3 \} & (k \geq 2) \end{cases}$$

Ref.

Pbm. Determine the least value f(k) which makes the following proposition true.

Prop. Every digraph D with $\delta^+(D) \geq \underline{f(k)}$ contains k vertex-disjoint dicycles.

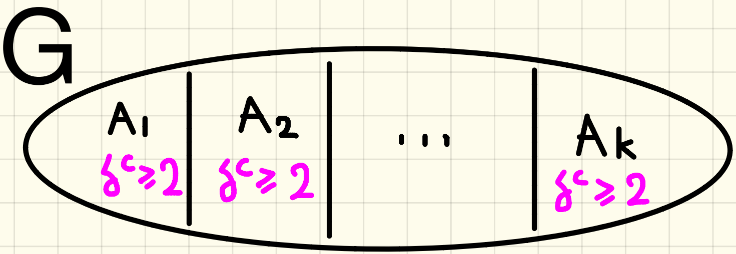
We obtained the following theorem.

Thm 1. (Ruonan Li, Guanghui Wang and F.)

Let G be an edge-colored graph with $\delta^c(G) \geq g(k)$.

Then G can be partitioned into k parts A1,...,Ak s.t.

$$\delta^c(G[A_i]) \geq 2 \text{ for } 1 \leq i \leq k.$$



Note: g(2) = 5.

Proof idea for Theorem 1.

In view of induction on k, we can check that proving the case k=2 is essential.

Thm. (Ruonan Li, Guanghui Wang, and F)

Let G be an edge-colored graph with $\delta^c(G) \geq 5$.

Then G can be partitioned into 2 parts A and B s.t.

$\delta^c(G[A]) \geq 2$ and $\delta^c(G[B]) \geq 2$.

It suffices to show that the following proposition is true.

Prop.1. If G is an edge-colored graph with $\delta^c(G) \geq 5$,

then G has two vertex-disjoint subgraphs A1,A2 s.t.

$$\delta^c(A_1) \geq 2 \text{ and } \delta^c(A_2) \geq 2.$$

- Prop.1 implies our theorem.

$\because$) Take A_1 and A_2 so that $|A_1 \cup A_2|$ is maximum.
Suppose $G-(A_1 \cup A_2) \neq \emptyset$. If $\delta^c(G-(A_1 \cup A_2)) \geq 2$, then
$[A_1, G-A_1]$ is a desired partition. But $\delta^c(G-(A_1 \cup A_2)) \leq 1$
would contradict the maximality of $|A_1 \cup A_2|$. $\square$

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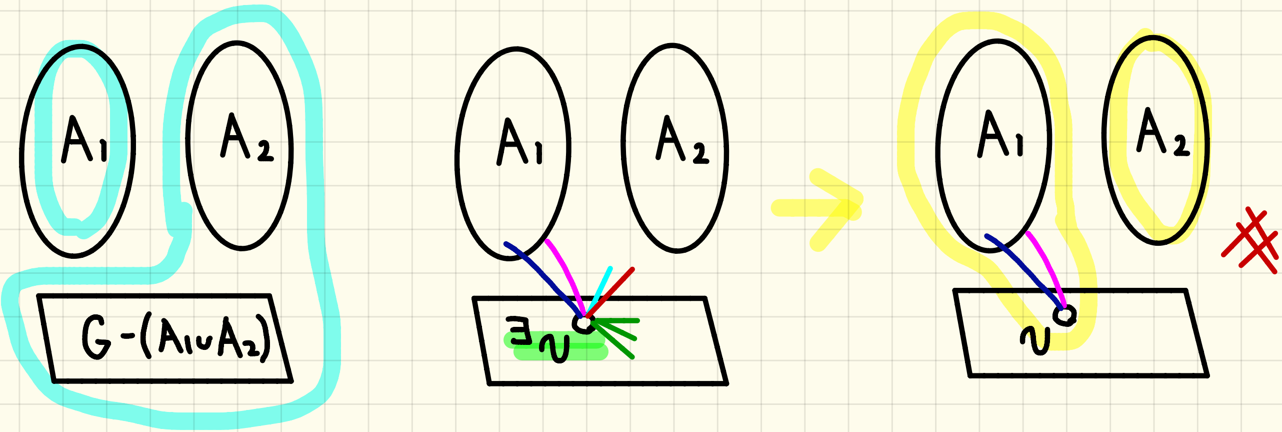


Proof ideas:

By contradiction, let G be a counterexample of Prop.1'.

We choose such an edge-colored G so that:

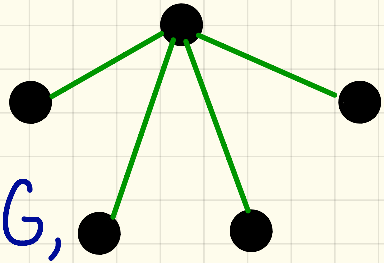
(i) $|G|$ is as small as possible, and subject to (i);

(ii) $|E(G)|$ is as small as possible, and subject to (ii);

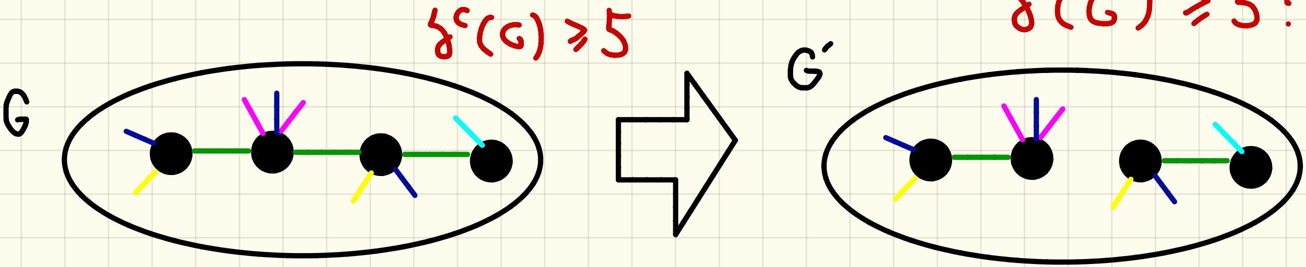
(iii) the number of colors in G is as large as possible.

By the choice of G, we see the following. For color j, let G_j be the subgraph of G obtained from color j edges.

Claim. Any G_j forms a star.

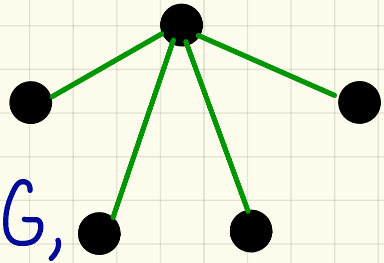


$\because$) If there is a mono. P_4 in G, then we can delete an edge from the P_4, which contradicts the choice of G.

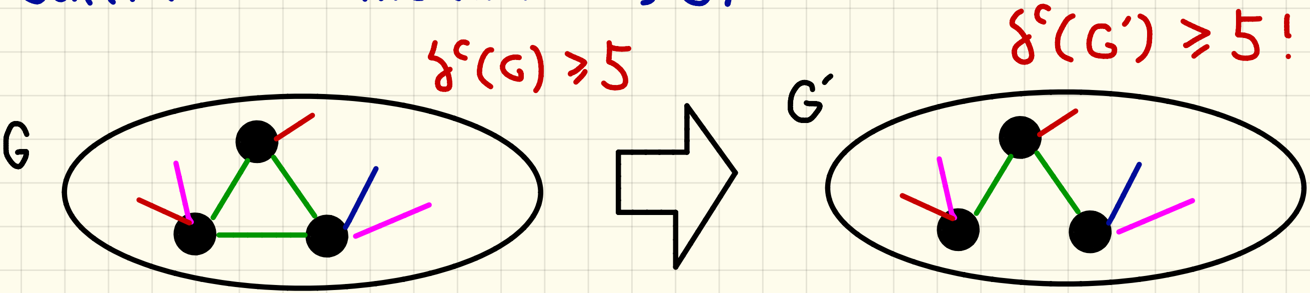


By the choice of G, we see the following. For color j, let Gj be the subgraph of G obtained from color j edges.

Claim. Any Gj forms a star.



Also, if there is a mono. C_3 in G, then we can delete an edge from the C_3, which contradicts the choice of G.



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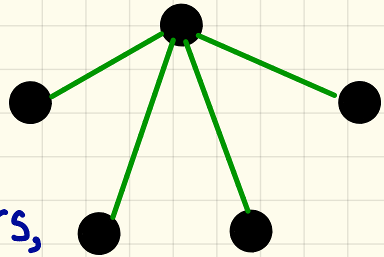
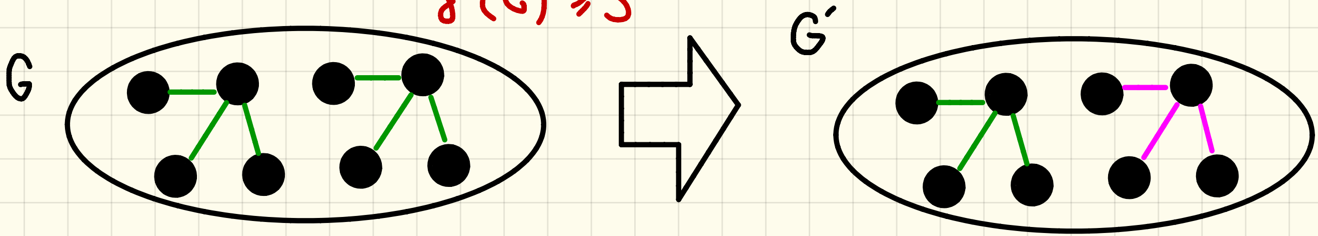


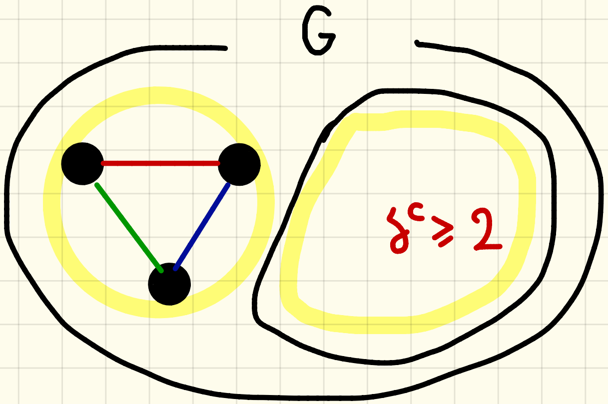
If there are two vertex-disj. mono. stars, then we can recolor one of them, which contradicts the choice of G. Thus, the claim works. $\square$

$\delta^c(G) \geq 5$ $\qquad$ $\delta^c(G') \geq 5$!



If G contains a rainbow triangle,
we can easily find a desired partition.

Thus we may assume that
G has no rainbow triangle.



We also use some inductive argument
such as vertex deletions and edge contractions.

Utilizing these techniques, we can get
a contradiction..

Returning to the statement of Thm.1, " $\delta^c(G) \geq g(k)$ ",

let's observe how digraph things are involved in our Pbm.

$$g(k) = \begin{cases} 2 & (k=1) \\ \max\{ f(k)+1, g(k-1)+3 \} & (k \geq 2) \end{cases}$$

Ref.

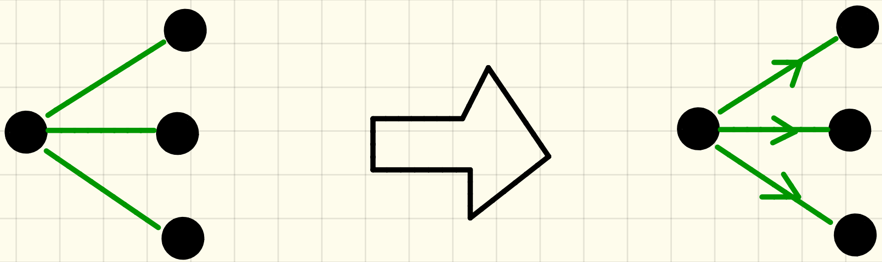
Pbm. Determine the least value f(k) which makes the following proposition true.

Prop. Every digraph D with $\delta^+(D) \geq \underline{f(k)}$ contains k vertex-disjoint dicycles.

Although the following argument is slightly different from the actual proof of our theorem, it'd be good to understand the proof approach (roughly).

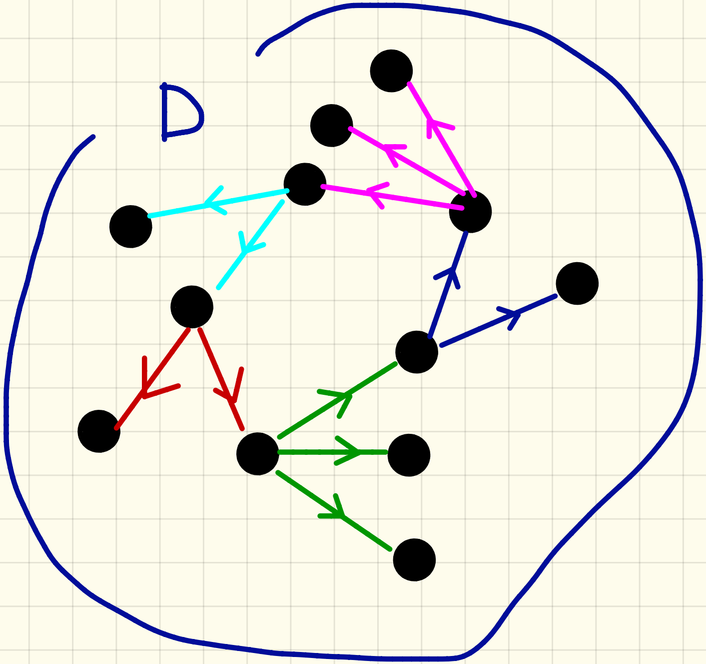
Recall the claim that any mono. component is a star.

From a mono. star, we can give an orientation on the edges in the following way:



Doing this way, we can construct a digraph D from G.

In view of Clm, we see that
any dicycle in D forms a properly colored cycle in G.



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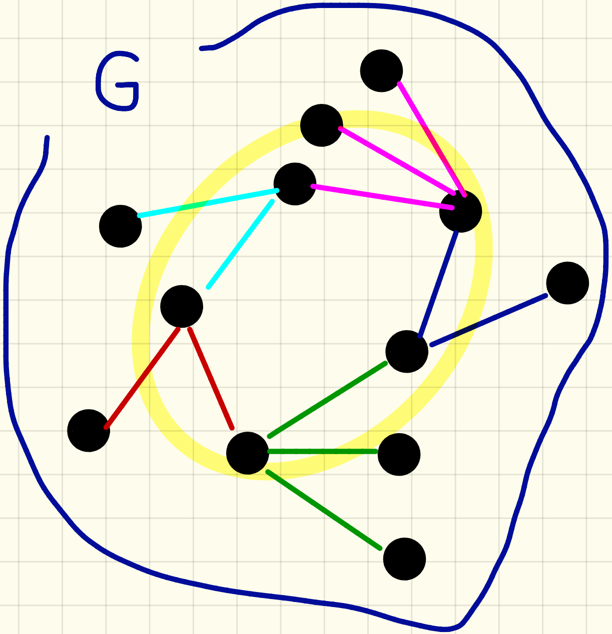


Thus, if $\delta^+(D) \geq f(k)$,
then we can find k vertex-disj.
properly colored cycles, and
hence we get a desired partition!

- Conj. is true for b=2 in edge-colored complete bip. graphs.

Cor. of Th.8 in Part I !

Thm 2. (Ruonan Li, Shenggui Zhang, and F)

Let G be an edge-colored complete bip. graph with

$$\delta^c(G) \geq a+2.$$

Then G can be partitioned into 2 parts A and B s.t.

$$\delta^c(G[A]) \geq a \text{ and } \delta^c(G[B]) \geq 2.$$

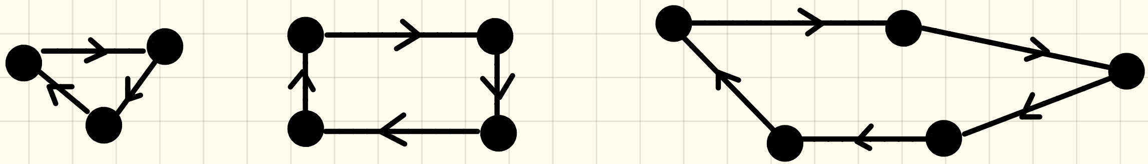
We also showed that our problem for the case b=2 has close links with Bermond-Thomassen's conjecture.

Pbm. Determine the least value f(k) which makes the following proposition true.

Prop. Every digraph D with $\delta^+(D) \geqslant f(k)$ contains k vertex-disjoint dicycles.

Conj. (Bermond and Thomassen, JGT'81)

$$f(k) = 2k-1.$$



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Prop. Every digraph D with $\delta^+(D) \geqslant f(k)$ contains k vertex-disjoint dicycles.

Conj. (Bermond and Thomassen, JGT'81)

$$f(k) = 2k-1.$$

Known results.

- true for k=1,2,3. (k=1,2: Thomassen '83; k=3: Lichiadopol et al. '09)
- $f(k) \leqq 64k$ (Alon, JCTB'97)

Thm 3. (Ruonan Li, Guanghui Wang, and F)

If our conjecture is true for b=2 then $f(k) \leqq 3k-1$.

Conj. (Bermond and Thomassen, JGT'81)

$$f(k) = 2k-1.$$

Known results.

- true for k=1,2,3.
- $f(k) \leqq 64k$ (Alon, JCTB'97)

Further results.

Thm 4. (Ruonan Li, Guanghui Wang and F)

Let G be an edge-colored complete graph with

$$\delta^c(G) \geq a+3.$$

Then G can be partitioned into 2 parts A and B s.t.

$$\delta^c(G[A]) \geq a \text{ and } \delta^c(G[B]) \geq 2.$$

Further results.

Thm 5. (Ruonan Li, Guanghui Wang, and F)

Let G be an edge-colored graph of order n with

$$a \geqslant b \geqslant 1 \text{ and } \delta^c(G) \geqslant 2\ln n + 4(a-1).$$

Then G can be partitioned into 2 parts A and B s.t.

$$\delta^c(G[A]) \geqslant a \quad \text{and} \quad \delta^c(G[B]) \geqslant b.$$